A Remark on Joint Probability Distributions for Quantum Observables

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A simple formal result concerning joint probability distributions in quantum mechanics is obtained. It is shown that some relatively weak properties of the joint distributions cannot be satisfied in the scope of standard quantum mechanics.

1. Justification

The question "Do there exist joint probability distributions for uncompatible quantum observables?" is commonly considered as ultimately solved by the appropriate theorem of v. Neumann [1]. Mere posing it is suspected now to be an assault on all sacrosancta of quantum mechanics. So a few words of justification seem to be desirable.

The remarkable amount of rigorous mathematical theorems [2] strenghtening the result of v. Neumann does not, in my opinion, make the problem of quantum joint distributions vain. For let us observe that any such "no go" theorem must at the very beginning define the notion of joint probability distribution in the quantum context. Thus the negative conclusion of such a theorem refers rather to the assumed properties of quantum joint distributions then to the existence of quantum joint distributions "in general". So one can defend the point of view that the theorems, allegedly precluding the existence of quantum joint distributions for uncompatible observables, indicate merely what the distributions should not be like provided they exist.

The presented paper gives new limitations on the possible properties of quantum joint distributions.

2. Definitions and Assumptions

Throughout this work the standard description of quantum systems [3] is accepted. We restrict ourself to the systems without superselection rules.

Notation: A_1, A_2, A: observables (self-adjoint linear operators on a Hilbert space $\mathscr{H}$ of dimension at least three); ψ: a unit vector of $\mathscr{H}$ (a pure state); $\mathscr{L}$: the set of all orthogonal projections on $\mathscr{H}$ endowed with the natural structure of an orthomodular σ-ortho-poset (the quantum logic); P_1, P_2, P, Q: elements of $\mathscr{L}$; E: the unit operator on $\mathscr{H}$; $\mathscr{B}(\underline{R})$: the algebra of Borel subsets of the real line $\underline{R}$; S_1, S_2, S: elements of $\mathscr{B}(\underline{R})$; $A(S)$: the projection corresponding to S via the spectral decomposition of A; $\langle 0, 1\rangle$: the unit interval.

The joint probability distribution for observables A_1, A_2 in a pure state ψ will be denoted by μ_{ψ, A_1, A_2}. We assume the following properties of joint probability distributions (for every A_1, A_2, ψ, S_1, S_2, S):

(i) μ_{ψ, A_1, A_2} is a probability measure on the real plane.

(ii) For given A_1, A_2, ψ there exists one and only one joint distribution μ_{ψ, A_1, A_2}.

(iii) For any real Borel functions f_1, f_2 on $\underline{R}$

$$\mu_{\psi, f_1(A_1), f_2(A_2)}(S_1 \times S_2) = \mu_{\psi, A_1, A_2}(f_1^{-1}(S_1) \times f_2^{-1}(S_2)) .$$

(iv) $\mu_{\psi, A_1, A_2}(S_1 \times S_2) = \mu_{\psi, A_2, A_1}(S_2 \times S_1)$.

(v) $\mu_{\psi, A_1, A_2}(S \times \underline{R}) = (\psi, A_1(S)\,\psi)$.

The above properties are rather weak and natural. The third one is well known as the correspondence rule of v. Neumann [4] used in his proof of non-existence of joint probability distributions for uncompatible observables. The fifth property above assures reasonable marginal distributions [5].

3. Only product distributions?

The joint probability distributions μ_{ψ, A_1, A_2} for arbitrarily fixed ψ and all pairs A_1, A_2 define the mapping

$$\mu_\psi\colon \mathscr{L} \times \mathscr{L} \mapsto \langle 0, 1\rangle$$

by

$$\mu_\psi(P_1, P_2) = \mu_{\psi, A_1, A_2}(S_1 \times S_2) \tag{1}$$

where $A_1(S_1) = P_1$, $A_2(S_2) = P_2$. The mapping μ_ψ is well defined owing to (ii) and (iii).

Now the property (iv) reads:

$$\mu_\psi(P_1, P_2) = \mu_\psi(P_2, P_1) \tag{2}$$

for every ψ, P_1, P_2. The property (v) takes the form:

$$\mu_\psi(P, E) = (\psi, P\,\psi) \tag{3}$$

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for every ψ, P. Taking into account the property (i) we obtain a stronger version of the last formula:

$$P_1 \psi = \psi \quad \text{implies} \quad \mu_\psi(P_1, P_2) = (\psi, P_2 \psi) \tag{4}$$

for every ψ, P_1, P_2.

Let us observe that if $P_1, P_2, \ldots$ is a countable family of mutually orthogonal elements of $\mathscr{L}$ ($P_i P_k = 0$ for $i \neq k$) then for every ψ, Q:

$$\mu_\psi(Q, \sum_n P_n) = \sum_n \mu_\psi(Q, P_n), \tag{5}$$

i.e. μ_ψ is a σ-additive function in both arguments. A straightforward proof of this fact rests on (i). Hence, if $Q\psi \neq 0$, then

$$\mu_{\psi, Q}(P) := \frac{\mu_\psi(Q, P)}{(\psi, Q \psi)} \tag{6}$$

turns out to be a probability measure on the logic $\mathscr{L}$ [6]. The celebrated theorem of Gleason [7] tells us now that $\mu_{\psi, Q}$ is uniquely related to a positive semidefinite, self-adjoint operator of trace one, say ϱ, such that

$$\mu_{\psi, Q}(P) = \operatorname{Tr} \varrho P \tag{7}$$

for every P (we have assumed $\mathscr{H}$ to be of at least three dimensions just in order to assure the applicability of the Gleason theorem. In the two-dimensional case the situation is really wild [8] and our reasoning does not hold true therein).

Let P_0 be the projection on the one-dimensional subspace of $\mathscr{H}$ spanned by ψ. Then

$$\mu_{\psi, Q}(P_0) = 1 \tag{8}$$

and the probability measure on $\mathscr{L}$ defined by (6) must be equal to the one associated with ψ by the standard rule of quantum mechanics:

$$\mu_{\psi, Q}(P) = \operatorname{Tr} P_0 P = (\psi, P \psi) \tag{9}$$

for every P. From (6) we get finally:

$$\mu_\psi(Q, P) = (\psi, Q \psi)(\psi, P \psi). \tag{10}$$

Observe, that this result is valid also for the case $Q\psi = 0$, since in this case $\mu_\psi(Q, P) = 0$.

So the apparently innocent properties of the quantum joint probability distributions that we have assumed, lead to hardly acceptable result: for any pair A_1, A_2 of observables and in any pure state ψ, the quantum joint probability distribution is equal to the product distribution:

$$\mu_{\psi, A_1, A_2}(S_1 \times S_2) = (\psi, A_1(S_1) \psi)(\psi, A_2(S_2) \psi). \tag{11}$$

We give below some argument for rejecting the product distributions, another one can be found in the paper of Margenau and Hill [9].

4. Compatibility

It is commonly accepted that a pair of compatible observables can be treated, to some extent, as a pair of classical stochastic variables, whereas the essentially quantum features appear for uncompatible observables. Two compatible self-adjoint operators A_1, A_2 have the well defined joint probability distribution in every state, viz. the distribution generated by

$$\mu_{\psi, A_1, A_2}(S_1 \times S_2) = (\psi, A_1(S_1) A_2(S_2) \psi). \tag{12}$$

This probability distribution clearly possesses the properties that we have assumed for the general case. If one agrees that the formula (12) provides the point probability distribution for the special case of compatible observables, then one must complete the list of assumed properties of μ_{ψ, A_1, A_2} by the following one:

(vi) For every S_1, S_2, if A_1 and A_2 are compatible, then

$$\mu_{\psi, A_1, A_2}(S_1 \times S_2) = (\psi, A_1(S_1) A_2(S_2) \psi), \tag{13}$$

or equivalently:

$$P_1 P_2 = P_2 P_1 \quad \text{implies}$$
$$\mu_\psi(P_1, P_2) = (\psi, P_1 P_2 \psi) \tag{14}$$

for every P_1, P_2, ψ.

Unfortunately, the product joint probability distributions does not obey this rule, as can be easily seen. Indeed, from (vi) we infer that

$$\mu_\psi(P, E - P) = 0,$$

whereas it follows from (11) that

$$\mu_\psi(P, E - P) = (\psi, P \psi)(1 - (\psi, P \psi)).$$

Comparing these two expressions we obtain:

$$(\psi, P \psi) = 0 \quad \text{or} \quad 1$$

which evidently does not hold true for arbitrary chosen P, ψ.

5. What does it mean?

The obtained result can be considered as a new strong evidence against the hypothesis of existence of joint probability distributions for uncompatible observables. It has an advantage to be independent from the suspicious second correspondence rule of v. Neumann [10] (see the mentioned paper of Margenau and Hill [11] for a critique), which is basic for other "impossibility" theorems for quantum joint distributions [12]. On the contrary, the assumed properties look innocent and plausible.

However, there is still the already mentioned second way to interpret such kind of results. Accepting this point of view one can ask which of the assumed properties must be rejected to save the existence of quantum joint probability distributions? Let us note that quantum mechanics implies a theory of probability of its own [13], which is a generalization of the classical (Kolmogorov's) theory. The crucial difference is the break down of the distributivity property of the underlying lattice of events. But this difference manifests itself exactly as an appearance of uncompatible observables in quantum mechanics. Consequently, joint probability distributions for uncompatible observables should be non-classical, hence the seemingly most natural property (i) should be rejected.

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